Self-similarity degree of deformed statistical ensembles

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A R T I C L E   I N F O

A B S T R A C T

We consider self-similar statistical ensembles with the phase space whose volume is invariant under the deformation that squeezes (expands) the coordinate and expands (squeezes) the momentum. The related probability distribution function is shown to possess a discrete symmetry with respect to manifold action of the Jackson derivative to be a homogeneous function with a self-similarity degree q fixed by the condition of invariance under $(n + 1)$-fold action of the related dilatation operator. In slightly deformed phase space, we find the homogeneous function is defined with the linear dependence at $n = 0$, whereas the self-similarity degree equals the gold mean at $n = 1$, and $q \to n$ in the limit $n \to \infty$. Dilatation of the homogeneous function is shown to decrease the self-similarity degree q at $n > 0$.

1. Introduction

Long-range interaction, long-time memory effects and evolution kinetics delayed into power law are known to facilitate the formation of multifractal phase space of complex systems whose investigations have stipulated a rise of interest in deformed thermodynamic systems [1–3]. A cornerstone of the related deformation procedure is formal replacement of the standard logarithm function in the Boltzmann entropy with some deformed version. Along this line, it is worthwhile to emphasize the Tsallis-type thermostatistics where distribution functions are characterized with power-law tails [4–10], and so-called basic-deformed statistical mechanics where probability distributions have a natural cut-off in the energy spectrum [11,12]. A specific peculiarity of such types of system is the self-similarity of related phase space whose volume is invariant under the deformation that combines the coordinate squeezing and the momentum expanding (or, vice versa, expanding the first axis and squeezing the second) [13].

Moreover, making use of a deformation procedure in quantum mechanics allows one to present non-trivially several physical fields from black holes to anyon superconductivity [14]. The formal basis of related quantum groups is so-called q-calculus [15–17], originally introduced by Heine and Jackson [18,19] to study the basic hypergeometric series [20]. From a mathematical point of view, the q-calculus represents the most suited formalism to investigate (multi)fractal sets whose generation is provided with manifold action of the dilatation operator related to the Jackson derivative [13]. It appears [21,22] for q-deformed bosons and fermions, a natural generalization of thermostatistics is based on the formalism of the q-calculus, whereas the stationary solution of a classical deformed Fokker–Planck equation represents a q-analogue of the exponential function in the framework of the basic hypergeometric series [11,23]. Related systems exhibit a discrete scale invariance whose description [24] is achieved by using the Jackson derivative and integral, generalizing the regular derivative and integral for discretely self-similar systems (for example, the free energy of spin systems on a hierarchical lattice is related to a homogeneous function being the q-integral [25]).

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Our consideration is devoted to determination of the exponent q that plays a central role as the non-extensivity parameter in the Tsallis thermostatistics and as the degree of the homogeneous function in the theory of self-similar systems.[1]Following Ref. [27] we consider in Section 2 the form of the escort probability distribution function based on the multiplicativity condition for this function related to a composite statistical system. In accordance with supposition [6], we show the escort probability distribution function takes the power-law form with the non-integer exponent reduced to the Tsallis non-extensivity parameter. Section 3 is devoted to a short presentation of main statements of the quantum group theory [26] to describe the symmetry of self-similar systems. Within standard formalism of the Lee groups [28], we show in Section 4 that invariance of the self-similar system under manifold action of the Jackson derivative derives an equation for the determination of the dependence of the self-similarity degree of the dilatation. Section 5 concludes our consideration with passage to the general case of affine transformation whose dilatation parameter changes at each step of transformation.

2. Non-extensivity parameter

The formalism of non-extensive thermostatistics is known to be based on the definition of a generalized logarithm [3]

$$\ln_q(x) := \frac{x^{1-q} - 1}{1 - q} \tag{1}$$

being deformed with non-extensivity parameter q. Making use of this logarithm in the entropy

$$S := \left\langle \ln_{2-q}(1/p) \right\rangle = -\sum_{i=1}^{W} \ln_{2-q}(p_i) p_i \tag{2}$$

accompanied with both normalization condition and definition of the internal energy E

$$\sum_{i=1}^{W} p_i = 1, \qquad \sum_{i=1}^{W} \varepsilon_i \mathcal{P}\big(f(p_i)\big) = E \tag{3}$$

arrives at the Tsallis thermostatistics. As usual, we suppose there to be statistical states scattered with probabilities p_i and use the escort distribution

$$\mathcal{P}\big(f(p_i)\big) := \frac{f(p_i)}{\sum_{i=1}^{W} f(p_i)}, \qquad \sum_{i=1}^{W} \mathcal{P}\big(f(p_i)\big) = 1 \tag{4}$$

to find physical observable value types of the internal energy E. Within the framework of the thermostatistics version [6], a function $f(p_i)$ is supposed to be of power-law:

$$f(p_i) \propto p_i^q. \tag{5}$$

It was shown recently [27] that the dependence (5) follows from the multiplicativity condition

$$f(p_{ij}^{AB}) = f(p_i^A) f(p_j^B) \tag{6}$$

for composite system AB consisting of non-dependent subsystems A and B whose probabilities are connected with the condition $p_{ij}^{AB} = p_i^A p_j^B$. Indeed, inserting into Eq. (6) the series

$$f(x) = x^\delta \sum_{n=0}^{\infty} C_n x^n \tag{7}$$

with unknown exponent $0 < \delta < 1$ and coefficients C_n, one derives the equation

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (C_m - \delta_{mn}) C_n \, (p_i^A)^m (p_j^B)^n = 0. \tag{8}$$

As it holds for arbitrary probabilities p_i^A, p_j^B, the condition $(C_m - \delta_{mn})C_n = 0$ must be fulfilled to arrive at equalities $C_m = C_n = 1$ for some fixed term n of the series (7), while $C_n = 0$ otherwise. As a result, we obtain the relation (5) with the exponent

$$q = n + \delta, \quad n = 0, 1, 2, \ldots \tag{9}$$

which is necessary to find. Within the framework of the assumption that the phase space of a non-extensive statistical system has a discrete symmetry under deformation, we will show further the escort probability defined by Eqs. (4) and (5) represents a homogeneous function with a self-similarity degree (9).

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[1] It is worthwhile to point out that the notion q has been introduced in the quantum group theory [26] for the deformation parameter. To avoid misunderstanding we will note this parameter as λ instead of q.

3. Description of self-similarity

In this section, we set forth the main statements of the theory of self-similar systems [29,30] and the quantum group theory [26] to be needed in the following. Along this line, our consideration is based on definition of the dilatation operator

$$D_x^\lambda := \lambda^{x\partial_x}, \qquad \partial_x \equiv \frac{\partial}{\partial x}. \tag{10}$$

By using a formal expansion in the Taylor series, its action on the power-law function gives

$$D_x^\lambda x^n = \lambda^{x\partial_x} x^n = \sum_{m=0}^{\infty} \frac{[\ln(\lambda)(x\partial_x)]^m}{m!} x^n = \sum_{m=0}^{\infty} \frac{(n\ln\lambda)^m}{m!} x^n = (\lambda x)^n. \tag{11}$$

Then, for an arbitrary analytical function $f(x)$ one obtains in a similar manner

$$D_x^\lambda f(x) = \sum_{m=0}^{\infty} \frac{\left[\partial_x f(x)|_{x=0}\right]^m}{m!} D_x^\lambda x^m = \sum_{m=0}^{\infty} \frac{\left[\partial_x f(x)|_{x=0}\right]^m}{m!} (\lambda x)^m = f(\lambda x). \tag{12}$$

With respect to a homogeneous function defined with the condition

$$h(\lambda x) := \lambda^q h(x) \tag{13}$$

action of the λ-dilatation operator (10) is described by the eigen value equation

$$D_\lambda h(x) = \lambda^q h(x). \tag{14}$$

On the other hand, the Jackson derivative

$$\mathcal{D}_x^\lambda := \frac{D_x^\lambda - 1}{(\lambda - 1)x}, \tag{15}$$

connected with the dilatation operator by means of the commutator $[\mathcal{D}_x^\lambda, x] = D_x^\lambda$, is characterized by the equality

$$\left(x\mathcal{D}_x^\lambda\right)h(x) = [q]_\lambda h(x). \tag{16}$$

Here, $h(x)$ is the homogeneous function subject to the self-similarity property (13) and the λ-basic number

$$[q]_\lambda := \frac{\lambda^q - 1}{\lambda - 1} \tag{17}$$

is introduced to be equal to the exponent q in the absence of dilatation ($\lambda = 1$) and to increase as λ^{q-1} in the limit $\lambda \to \infty$. According to Eqs. (14) and (16) the homogeneous function $h(x)$ is the eigen function of both λ-dilatation operator D_λ and the λ-derivative $x\mathcal{D}_x^\lambda$ whose eigen-values are the power-law function λ^q and the λ-basic number $[q]_\lambda$ given by Eq. (17), respectively.

By using the λ-deformed Leibnitz rule for some functions $f(x)$ and $g(x)$ [20]

$$\begin{aligned}
\mathcal{D}_x^\lambda f(x)g(x) &= g(x)\mathcal{D}_x^\lambda f(x) + f(\lambda x)\mathcal{D}_x^\lambda g(x) \\
&= g(\lambda x)\mathcal{D}_x^\lambda f(x) + f(x)\mathcal{D}_x^\lambda g(x),
\end{aligned} \tag{18}$$

it is easy to be convinced that the homogeneous function, being the solution of the eigen equation (16), has the form

$$h(x) = A_\lambda(x)x^q \tag{19}$$

to be determined with a degree $q > 0$ and an amplitude $A_\lambda(x)$ obeying the condition

$$\mathcal{D}_x^\lambda A_\lambda(x) = 0. \tag{20}$$

From here, with accounting for Eqs. (15) and (12) one obtains the λ-periodicity property

$$A_\lambda(\lambda x) = A_\lambda(x). \tag{21}$$

This property is easily shown to be satisfied with the series

$$A(x) = x^{-q} \sum_{m=-\infty}^{\infty} p(\lambda^m x)\lambda^{-qm} \tag{22}$$

where $p(x)$ is arbitrary periodic function vanishing at $x = 0$ together with its first $n + 1$ derivatives where $n \equiv [q]$ is an integer of q. According to Eq. (22) the amplitude of the homogeneous function (19) is periodical in $\ln x$ with period $\ln \lambda$ [24]. In the simplest case $p(x) = 1 - \cos(x)$, one obtains the Weierstrass–Mandelbrot function

$$h(x) = \sum_{m=-\infty}^{\infty} \frac{1 - \cos(\lambda^m x)}{\lambda^{qm}} \tag{23}$$

whose graph is a fractal set with the dimension $2 - q$ [30]. Introducing the self-similarity exponent in complex form

$$q_m \equiv q + i\frac{2\pi}{\ln \lambda} m, \quad m = 0, \pm 1, \pm 2, \dots, \tag{24}$$

one represents the homogeneous function (19) as the Melline series

$$h(x) = \sum_{m=-\infty}^{\infty} A_m x^{q_m} \tag{25}$$

to be the formal hallmark of the self-similarity. However, it is much more convenient physically to use the Fourier representation [31]

$$A_\lambda(x) = \sum_{m=-\infty}^{\infty} A_m \exp\left(i\frac{2\pi}{\ln \lambda} m \ln x\right) \simeq A_0 + 2A_1 \cos\left(2\pi \frac{\ln x}{\ln \lambda}\right) \tag{26}$$

where the last estimation takes into account that the coefficients $A_m = A_{-m}$ decay very fast with increasing m. Logarithmically oscillating behavior is known [33] to relate to increase of levels number of the hierarchical tree which describes a dilatation cascade under manifold deformation of the self-similar system.

Thus, the above formalism describes properties of statistical systems possessing invariance under discrete deformations with a finite scale [31,32]. The phase space of such systems is known to form a fractal set whose generation is provided with manifold action of the dilatation operator (10) that expands ($\lambda > 1$) or squeezes ($\lambda < 1$) the x axis with the finite scale λ. Along this line, the set $\{h(x)\}$ of homogeneous functions (19) forms a vector space which provides the Fock-Bargmann representation $\{|\psi\rangle\}$ of the Weyl–Heisenberg algebra generated by the set of creation and annihilation operators $\hat{a}$, a accompanied with the particle number operator $N \equiv \hat{a}a$ to be obeyed the commutation relations [26,34]

$$[a, \hat{a}] = 1, \qquad [N, \hat{a}] = \hat{a}, \qquad [a, N] = a. \tag{27}$$

The passage from the Hilbert space $\{|\psi\rangle\}$ to the functional one $\{h(x)\}$ is provided with the following identification:

$$\hat{a} \to x, \qquad a \to \partial_x, \qquad N \to x\partial_x. \tag{28}$$

Remarkably, the homogeneous functions are marked out among the whole set of functions in a similar manner as coherent states make the same among all possible quantum states [17,13]. In fact, the wave function related to the coherent states

$$|x\rangle := \exp\left(x\hat{a} - \bar{x}a\right)|0\rangle = \exp\left(-\frac{|x|^2}{2}\right) \sum_{n=0}^{\infty} \frac{(x\hat{a})^n}{n!}|0\rangle$$

$$= \exp\left(-\frac{|x|^2}{2}\right) \sum_{n=0}^{\infty} h_n(x)|n\rangle \tag{29}$$

is reduced to expansion over eigenkets $|n\rangle = \frac{1}{\sqrt{n!}}\hat{a}^n|0\rangle$ with coefficients $h_n(x) \equiv h(x)|_{q=n}$ being the homogeneous function (19) with the integer self-similarity degree $q = n$. In accordance with Eq. (12), one obtains the deformed coherent state

$$D_x^\lambda |x\rangle = |\lambda x\rangle = \exp\left(-\frac{|\lambda x|^2}{2}\right) \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{\sqrt{n!}}|n\rangle \tag{30}$$

whose comparison with the last of expressions (29) shows that, with accuracy in the non-essential multiplier $1/\sqrt{n!}$, the deformed homogeneous function

$$\tilde{h}_n(x) := D_x^\lambda h_n(x) = \frac{(\lambda x)^n}{\sqrt{n!}} \tag{31}$$

related to the fractal set differs from the non-deformed one by the factor λ^n being inherent in the squeezing transformation in the course of the fractal generation [30]. Thus, we can conclude that manifold dilatation of the coherent state (29) relates to the fractal iteration process.

With deformation of the Weyl–Heisenberg algebra (27), the operator a in relations (28) must be replaced with the Jackson derivative (15). Moreover, similar to quantum optics, it is convenient to introduce the squeezing operator [17,13]

$$S := \sqrt{\lambda}\, D_x^\lambda = \sqrt{\lambda}\, e^{(\ln \lambda)N}, \quad N \equiv x\partial_x \tag{32}$$

that picks out the factor $\sqrt{\lambda}$ in front of the dilatation operator (10). At real x values, the coordinate x and the momentum $p = -i\partial_x$ are associated with the annihilation and creation operators

$$c := \frac{1}{\sqrt{2}}\left(\hat{a} + a\right), \qquad c^\dagger := \frac{1}{\sqrt{2}}\left(\hat{a} - a\right) \tag{33}$$

being hermitian conjugated. At the squeezing, these operators transform as follows [13]

$$\tilde{c} := S^{-1}cS = \frac{1}{\sqrt{2}}\left(\lambda^{-1}\hat{a} + \lambda a\right) = \cosh(\ln \lambda)c - \sinh(\ln \lambda)c^\dagger,$$
$$\tilde{c}^\dagger := S^{-1}c^\dagger S = \frac{1}{\sqrt{2}}\left(\lambda^{-1}\hat{a} - \lambda a\right) = \cosh(\ln \lambda)c^\dagger - \sinh(\ln \lambda)c. \tag{34}$$

Above relations show the squeezing transformation is reduced to the Bogoliubov rotation with the angle $\ln \lambda$ in a plane spanned by the operators c and $c^\dagger$. This means that the related transformations of the pair of conjugated coordinate and momentum

$$\tilde{x} := S^{-1}xS = \lambda^{-1}x, \qquad \tilde{p} := S^{-1}pS = \lambda p \tag{35}$$

keep the phase space volume to be invariant under the squeezing.

The measure of a fractal set generated by the dilatation transformation [24]

$$M(\ell_m, R) = A_\lambda(\ell_m, R)\ell_m^d(\ell_m/R)^{-d_f} \tag{36}$$

depends in the power form of the linear size $\ell_m = \lambda^m$ of covering balls related to m-th step of the fractal generation. (Here, $A_\lambda(\ell_m, R)$ is an oscillation amplitude type of $A_\lambda(x)$ in the homogeneous function (19), R is characteristic size of a fractal region of dimension d_f embedded in a space with topological dimension $d \geq d_f$.) According to Eq. (12) the dilatation (10) transforms a box of size ℓ_m into the same of the size $\ell_{m+1} = \lambda\ell_m$, so that one has

$$D_\ell^\lambda M(\ell_m, R) = M(\ell_{m+1}, R) = \lambda^{d-d_f}M(\ell_m, R). \tag{37}$$

In the limit $m \to \infty$, the measure $M(R) := \lim_{m\to\infty} M(\ell_m, R)$ depends only on the characteristic fractal size R, for which the operator D_R^λ gives the eigen value equation

$$D_R^\lambda M(R) = R^{d_f} M(R). \tag{38}$$

Thus, the generator (10) of the dilatation group allows one to find the difference $d - d_f$ between topological and fractal dimensions – if the scale ℓ is dilated, and fractal dimension d_f itself – at dilating the fractal size R. On the other hand, making use of combinations $\ell\mathcal{D}_\ell^\lambda$ and $R\mathcal{D}_R^\lambda$ with the Jackson derivative (15) arrives at the λ-basic numbers $[d - d_f]_\lambda$ and $[d_f]_\lambda$, respectively.

In more complicated cases, when a complex system has a multifractal space [31,35], its measure is characterized by the partition function

$$Z_\lambda(\ell, R; q) := \sum_i^{N(\ell,R)} \ell^{\alpha q} \tag{39}$$

where summation is carried out over all balls i whose number $N := \ell^{-f(\alpha)}$ is determined by a spectral function $f(\alpha)$ with the argument being a singularity strength α [36]. Similar to Eq. (36), one obtains the homogeneous function

$$Z_\lambda(\ell, R; q) = A_\lambda(\ell, R; q)(\ell/R)^{\tau(q)} \tag{40}$$

where $A_\lambda(\ell, R; q)$ is an oscillating amplitude and $\tau(q)$ is the mass exponent defined by the equation

$$\tau(q) = \alpha_q q - f(\alpha_q) \tag{41}$$

with the specific singularity strength α_q being fixed by the conditions of the steepest-descent method:

$$\left.\frac{df}{d\alpha}\right|_{\alpha=\alpha_q} = q, \qquad \left.\frac{d^2f}{d\alpha^2}\right|_{\alpha=\alpha_q} < 0. \tag{42}$$

As a result, the equality type of (38)

$$D_\ell^\lambda Z_\lambda(\ell, R; q) = \lambda^{\tau(q)}Z_\lambda(\ell, R; q) \tag{43}$$

determines the mass exponent $\tau(q)$. Analogously, making use of the differentiation operator $\ell\mathcal{D}_\ell^\lambda$ gives the λ-basic number $[\tau(q)]_\lambda$.

4. Determining the self-similarity degree

Within the Lee group theory [28], the operator $x\mathcal{D}_x^\lambda$ appears as a generator of the transformation

$$T_x^\lambda(t) := \exp[t(x\mathcal{D}_x^\lambda)] \tag{44}$$

related to a continuous parameter t. By accounting for Eq. (16), the formal expansion of the exponential operator (44) in the Taylor series gives

$$T_x^\lambda(t)h(x) = \sum_{m=0}^\infty \frac{t^m(x\mathcal{D}_x^\lambda)^m}{m!}h(x) = \sum_{m=0}^\infty \frac{([q]_\lambda t)^m}{m!}h(x) = e^{[q]_\lambda t}h(x). \tag{45}$$

Thus, the group operator (44) has the eigen value $e^{[q]_\lambda t}$ determined by the λ-basic number (17) and the eigen function being the homogeneous function (19).

Let us introduce now the definition of the Lee group transformation

$$\mathcal{T}_x^\lambda(t) := \exp(t\mathcal{D}_x^\lambda) \tag{46}$$

being slightly different from Eq. (44) because the Jackson derivative itself $\mathcal{D}_x^\lambda$ is taken as the generator instead of the combination $x\mathcal{D}_x^\lambda$. Our main approach is based on the proposition that the self-similar system is invariant under the $(n+1)$-fold action of the λ-differentiation operator[2]

$$\left(\mathcal{D}_x^\lambda\right)^{n+1}h(x) := x^{-(n+1)}h(x). \tag{47}$$

Inserting here related homogeneous function (19), by accounting for Eq. (20) one obtains

$$\left(\mathcal{D}_x^\lambda\right)^{n+1}x^q = \left(\prod_{m=0}^n [q-m]_\lambda\right)x^{q-(n+1)}. \tag{48}$$

Then, the self-similarity condition (47) gives the transcendental equation

$$\prod_{m=0}^n [q-m]_\lambda = 1. \tag{49}$$

As a result, similar to Eqs. (45), the action of the dilatation group operator (46) is defined with chain of the following equalities:

$$\begin{aligned}
\mathcal{T}_x^\lambda(t)h(x) &= \left[1 + \sum_{m=0}^\infty \frac{t^{m+1}(\mathcal{D}_x^\lambda)^{m+1}}{(m+1)!}\right]h(x) \\
&= \left[1 + \sum_{m=0}^\infty \frac{\prod_{l=0}^m [q-l]_\lambda\, t^{m+1}x^{-(m+1)}}{(m+1)!}\right]h(x) = \sum_{m=0}^\infty \frac{(t/x)^m}{m!}h(x) = e^{t/x}h(x).
\end{aligned} \tag{50}$$

Comparison of Eqs. (45) and (50) at $t = 1$ shows that the generator $x\mathcal{D}_x^\lambda$ creates the exponential eigen value with the basic number $[q]_\lambda$, whereas making use of the Jackson derivative $\mathcal{D}_x^\lambda$, accompanied with the self-similarity condition (49), makes the exponent $1/x$ to be the inverse argument of the homogeneous function. Physically, the result (50) means that the dilatation strengthens the power-type function (19) exponentially within the domain of small values x. Principally important is that this property is provided with the self-similarity condition (49) only.

As a result, one needs to study when this condition is fulfilled. In slightly dilated system ($\lambda \to 1$) the condition (49) takes the simple form

$$\prod_{m=0}^n (q-m) = 1. \tag{51}$$

In the case of one-fold dilatation ($n = 0$), the self-similarity degree $q_0 = 1$ relates to the linear function (19). A much more interesting situation is realized at the two-fold deformation ($n = 1$), when the self-similarity degree is reduced to *the gold mean*

$$q_1 = \frac{1+\sqrt{5}}{2} \simeq 1.618. \tag{52}$$

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[2] It is interesting to point out that a periodic function $p(x)$ in the series (22) has as well $n + 1$ first derivatives vanishing in the point $x = 0$.
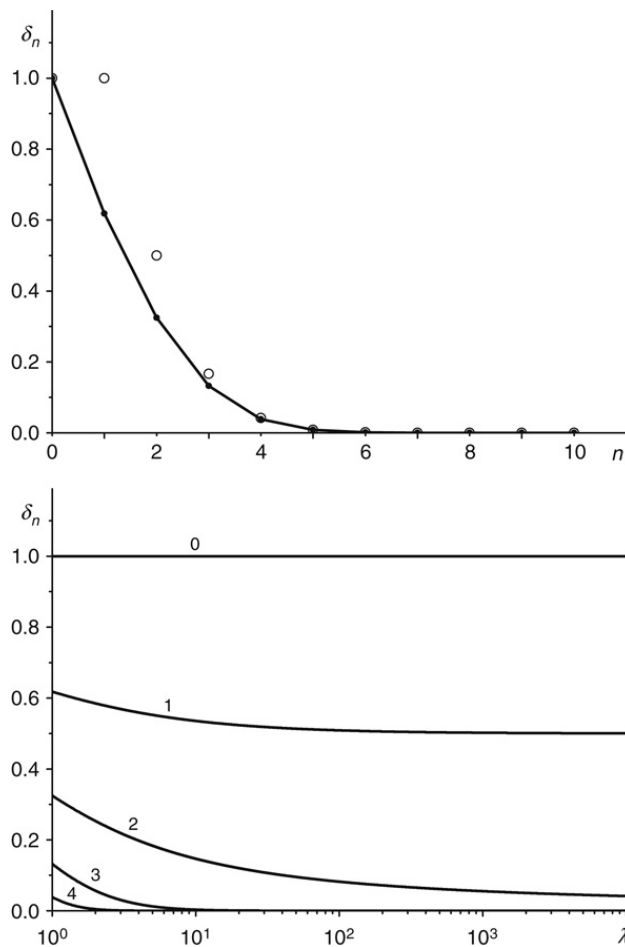
Fig. 1. Top: Dependence of the relative self-similarity degree $\delta_n = q_n - n$ on the number $n + 1$ of the dilatation steps in slightly dilated system, $\lambda \to 1$ [points connected with a solid line show numerical solutions of Eq. (51), open cycles relate to analytical solution (54)]; bottom: the same in dependence of the dilatation parameter λ at different integers n being pointed out near related curves.

For dilatation orders $n > 1$, one obtains more complicated expressions, which is convenient to present in graphical form (see Fig. 1a). This figure shows the self-similarity degree can be expressed as the binomial (9) that defines the non-extensivity parameter with fractional addition $0 < \delta < 1$. Within the first order of accuracy over δ, Eq. (51) takes the simple form

$$n!\delta_n \approx 1. \tag{53}$$

As a result, the self-similarity degree of the slightly dilated system reads

$$q_n \approx n + \frac{1}{n!}. \tag{54}$$

Fig. 1a shows this expression at $n = 1$ gives maximum error 47% which falls down very quickly achieving values less than 1% at $n > 3$.

In general case of the λ-dilated systems, the combination of Eqs. (49), (17) and (9) arrives at the equality

$$\delta_n \approx \frac{1}{\ln \lambda} \ln \left(1 + \frac{\lambda - 1}{[n]_\lambda!} \right) \tag{55}$$

and the self-similarity degree takes the estimation

$$q_n \approx n + \frac{1}{\ln \lambda} \ln \left(1 + \frac{\lambda - 1}{[n]_\lambda!} \right). \tag{56}$$

Corresponding values $\delta_n = q_n - n$ at different positive integers n are shown in Fig. 1b. At $n = 0$, the self-similarity degree $q_0 = 1$ does not depend on the dilatation λ, whereas with $n > 0$ increasing the q_n value monotonically decays the faster the more the dilatation order n. At that, the expression (56) gives an error of not more than 1% at $n > 3$.

5. Discussion

Above consideration concerns the properties of the self-similar statistical ensembles whose phase space has invariant volume since related deformations (35) combine coordinate squeezing with equal momentum expanding (and vice versa). According to Eq. (12), the mathematical tool to describe the deformation is based on the definition (10) of the dilatation operator accompanied with the Jackson derivative (15). In addition, the probability distribution function is reduced to the homogeneous function (19) being the eigen function of both dilatation operator and Jackson derivative whose eigen-values λ^q and $[q]_\lambda$ are determined by the self-similarity degree q. On the other hand, statistical states of a self-similar ensemble form a (multi)fractal set whose generation relates to manifold dilatations (31) of the homogeneous function with the integer self-similarity degree. As shown Eqs. (38) and (43), making use of the Jackson derivative allows for one to find the fractal dimension d_f as well as the mass exponent $\tau(q)$ of corresponding fractal and multifractal sets.

As shown in the comparison of Sections 2 and 3, the non-extensivity parameter (9) of the Tsallis thermostatistics coincides with the self-similarity degree q of the homogeneous function (19). Being the probability distribution, this function is supposed under the manifold action (47) of the Jackson derivative to be multiplied by the trivial power-law factor without the product (49) being put to equal one. Then, the action (50) of the Lee group operator (46) picks out the exponential factor which sharpens the power-law form of the homogeneous function within the domain of small argument values. As a result, we derive the Eq. (49) which defines the dependence of the self-similarity degree q on the dilatation λ at different numbers $n + 1$ of deformation iterations.

In the limit $\lambda \to 1$ related to the slightly deformed systems, we find the homogeneous function (19) is defined with the linear dependence ($q_0 = 1$) at the one-fold deformation ($n = 0$). A principle different case takes place at two-fold deformation ($n = 1$) when the self-similarity degree is reduced to the gold mean (52) being a hallmark of self-similarity [33]. According to the estimation (54), at $n > 3$, the degree q_n is slightly different of its integer $[q]$ being equal the number n.

As shown in Fig. 1b, a finite dilatation $\lambda \neq 1$ of a self-similar statistical ensemble keeps the linear form of the homogeneous function invariant at the one-fold deformation ($n = 0$). An increase of the number $n > 0$ arrives at dependencies $q_n(\lambda)$ that decay logarithmically slowly with the dilatation growth. In the limit $n \to \infty$, this decaying is estimated with the relation (56).

Touching on the physical meaning of the results obtained, we emphasize that above used self-similarity transformations are characterized with unique dilatation parameter λ being constant at all steps of the transformations. In the general case of *affine* transformations, one needs to consider a set of dilatation parameters $\{\lambda_m\}$ which can change at each step m of transformations. Moreover, in statistical systems, each of these transformations gets a specific weight $\alpha_m \geq 0$ normalized with the condition $\sum_m \alpha_m = 1$. As a result, the dilatation operator (10) takes the form

$$D_x := \sum_{m=-\infty}^{\infty} \alpha_m D_x^{\lambda_m} = \sum_{m=-\infty}^{\infty} \alpha_m \lambda_m^{x\partial_x} \tag{57}$$

to be related to the affine transformation

$$D_x f(x) = \sum_{m=-\infty}^{\infty} \alpha_m f(\lambda_m x) \tag{58}$$

instead of the homogeneous dilatation (12).

Consideration of self-affine systems whose observables are invariant under transformations (58) shows the inverse Melline transformation of coordinate dependence of such an observable has poles on the complex plane z at condition (see [37], Appendix in the last of Refs. [31,32] or Section 3.2 in Ref. [33])

$$\sum_{m=-\infty}^{\infty} \alpha_m^2 \lambda_m^{d_f+iz} = 1. \tag{59}$$

These poles are ranged in rows parallel with the real axis to be located within the band $1 - d_f \leq \Im z \leq 0$ whose width is fixed by the fractal dimension d_f of self-affine set obtained. The solutions of the Eq. (59)

$$z_n^m = -i\zeta_n + \frac{2\pi}{\ln \lambda_n} m, \quad m = 0, \pm 1, \pm 2, \ldots \tag{60}$$

are determined with set of parameters $\lambda_n \geq 1$ and $0 \leq \zeta_n \leq d_f$ related to the pole rows $n = 0, 1, \ldots, \nu$ (integer m defines the number of pole in given row). Respectively, the amplitude (22) of the homogeneous function (19) takes the form

$$A(x) = \sum_{n=0}^{\nu} x^{-\zeta_n} p_n \left(\frac{\ln x}{\ln \lambda_n} \right). \tag{61}$$

In the difference of self-similar systems, the amplitude (61) of self-affine homogeneous function does not only oscillate logarithmically, but decays monotonically with increasing argument if the number of pole rows is $\nu + 1 > 1$.

In the trivial case of simple systems, the phase space is smooth so that the fractal dimension $d_f = 1$ and all poles (60) are pure real. In this case, the values $\alpha_m = \lambda_m^{-1} = N^{-1}$ are determined by total number $N \to \infty$ of steps in the fractal generating, and above poles read

$$z_0^m = \frac{2\pi}{\ln N}m, \quad m = 0, \pm 1, \pm 2, \dots. \tag{62}$$

Then, the homogeneous function (19) is reduced to the linear power law ($q = 1$) with logarithmically periodical amplitude

$$p_0(x) = \sum_{m=-\infty}^{\infty} \alpha_m p_0\left(m\frac{\ln x}{\ln N}\right). \tag{63}$$

As pointed out in the Introduction, long-range interaction and long-time memory effects inherent in complex systems facilitate the formation of fractal phase space with dimension $d_f < 1$. Strengthening these effects decreases the fractal dimension to arrive at creation and consequent increase of imaginary pole rows in the points (60) with $n > 0$. One can suppose that these poles relate to different numbers n of manifold deformation (47) that determines the self-similarity degree q_n in dependence of the dilatation. In fact, we have shown above the simple systems are characterized by one-fold deformation ($n = 0$) to be determined by the linear homogeneous function with the self-similarity degree $q_0 = 1$ which corresponds to the real row of poles (62) with the logarithmically oscillating amplitude (63). The passage to the complex systems caused by long-range interaction and long-time memory arrives at the creation of the first row of imaginary poles (60). These poles are related to the two-fold deformation ($n = 1$) characterizing the Tsallis non-extensive thermostatistics with the self-similarity degree $q_1(\lambda)$ whose value decays logarithmically slowly with dilatation growth from the gold mean $q_1(1) \simeq 1.618$ to the minimum value $q_1(\infty) = 1$. As the amplitude (61) decays with the argument growth, the homogeneous function (19) related to the Tsallis thermostatistics is written in the form

$$h(x) = p_0 x + p_1\left(\frac{\ln x}{\ln \lambda_1}\right)x^Q, \quad Q \equiv q_1(\lambda_1) - \zeta_1. \tag{64}$$

Here, we take into account that the lower degree $q_0 = 1$ and in the limit $N \to \infty$ the row (62) is reduced to the single pole $z_0^0 = 0$ to transform the logarithmically periodical amplitude (63) into a constant p_0 related to the limit dilatation $\lambda_0 = N \to \infty$.

The equalities (64) represent main finding of our investigation. It means the homogeneous function of self-affine systems, being invariant under both infinite $\lambda_0 \to \infty$ and finite $\lambda_1 \neq 1$ dilatations, is reduced to the usual linear term accompanied with non-linear logarithmically oscillating addition. The self-similarity degree $Q = Q(\lambda_1)$ of this function is limited above by the gold mean to decay monotonically with the dilatation λ_1 increasing.

Finally, note that the study of extremely complex systems whose behavior is governed with more than one rows of imaginary poles (60) is beyond the scope of our consideration.

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